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Differential Forms in Synthetic Differential Supergeometry

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We present a basic theory of differential forms in synthetic differential supergeometry. Exterior differential calculus is developed, and Cartan's three magic formulas as well as a variant of de Rham's theorem are established.

INTRODUCTION

Supergeometry is a fascinating subject to both physicists and mathematicians. It has a close relationship with both algebraic geometry and differential geometry. It is expected to play a central role in any possible unification of relativity and quantum theory. For textbooks on supergeometry, the reader is referred to Bartocci *et al.* (1991) and DeWitt (1984).

Synthetic differential geometry lies between algebraic geometry and differential geometry, borrowing many important concepts and methods from the former, while trying to extend the scope of the latter. It is expected to cater to the pure science of spacetime of the 21th century. For textbooks on synthetic differential geometry, the reader is referred to Kock (1981), Lavendhomme (1996), and Moerdijk and Reyes (1991).

As far as we know, Nishimura (1998) and Yetter (1988) are the only two attempts to extend synthetic differential geometry to supergeometry. Both deal with a superization of the synthetic theory of vector fields, but go no further. The principal objective of this paper is to extend the scope of synthetic supergeometry to the theory of exterior differential calculus. Since supergravity theories have been formulated successfully in terms of differential forms, such an extension is of considerable interest not only to mathematicians, but also to mathematical physicists.

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In Section 1 we superize the notion of differential form in two ways and establish their equivalence. We establish two of the three superized Cartan formulas connecting Lie derivatives and interior products. In Section 2 we establish the existence of superized exterior differentiation and prove the remaining one of the three superized Cartan magic formulas. The last section is devoted to a version of de Rham's theorem.

This paper is to be regarded as a sequel to Nishimura (1998), but we will often omit the use of prefix "super." The field of integers mod 2, consisting of $\mathbf{0}$ ($= 0 \bmod 2$) and $\mathbf{1}$ ($= 1 \bmod 2$), is denoted by $\mathbb{Z}_2$. For any $\mathbf{p} \in \mathbb{Z}_2$, $(-1)^{\mathbf{p}}$ denotes 1 or -1 as $p = \mathbf{0}$ or $p = \mathbf{1}$. Sometimes an integer is regarded implicitly as an element of $\mathbb{Z}_2$, but the context should prevent any confusion. Given $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$, the canonical injection of $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ into $D(\mathbf{0}, \mathbf{1})^n$ and the canonical projection of $D(\mathbf{0}, \mathbf{1})^n$ onto $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ are denoted by $\iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n}$ and $\pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n}$, respectively. The space $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ is sometimes denoted by $D^{(\mathbf{p}_1,\ldots,\mathbf{p}_n)}$. We arbitrarily choose a microlinear space M ("microlinear" in the super context) once and for all. Familiarity with Lavendhomme (1996) up to Chapter 4 will be highly helpful.

1. DIFFERENTIAL FORMS

Given $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$, an *n-microcube of type* $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ *on* M is a function from $D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$ to M. We denote by $\mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ the totality of n-microcubes of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M. We denote by $\mathrm{T}^n M$ the set-theoretic union of $\mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}$ for all $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$. Given $\gamma \in \mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $a \in \mathbb{R}_e$, an n-microcube $\gamma \underset{i}{;} a = a \underset{i}{;} \gamma$ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M $(1 \leq i \leq n)$ is defined by

(1.1) $(\gamma \underset{i}{;} a)(d_1, \ldots, d_n = \gamma(d_1, \ldots, ad_i, \ldots, d_n)$
(1.2) $(a \underset{i}{;} \gamma)(d_1, \ldots, d_n = \gamma(d_1, \ldots, d_i a, \ldots, d_n)$

for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \ldots \times D^{\mathbf{p}_n}$. Given $\gamma \in \mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $a \in \mathbb{R}_o$, n-microcubes $\gamma \underset{i}{;} a$ and $a \underset{i}{;} \gamma$ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_i + 1, \ldots, \mathbf{p}_n)$ on $M (1 \leq i \leq n)$ are defined by (1.1) and (1.2), respectively, in which we should note that $\gamma \underset{i}{;} a \neq a \underset{i}{;} \gamma$ generally. Given $\gamma \in \mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and $\sigma \in \mathfrak{Perm}_n$, an n-microcube $\Sigma_\sigma(\gamma)$ of type $(\mathbf{p}_{\sigma^{-1}(1)}, \ldots, \mathbf{p}_{\sigma^{-1}(n)})$ on M is defined as follows:

(1.3) $\Sigma_\sigma(\gamma)(d_1, \ldots, d_n) = \gamma(d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_{\sigma^{-1}(1)}} \times \ldots \times D^{\mathbf{p}_{\sigma^{-1}(n)}}$.

A *differential n-form on M* is a mapping ω from $\mathrm{T}^n M$ to $\mathbb{R}$ satisfying the following conditions:

(1.4) $\omega(\gamma \underset{i}{;} a) = \omega(a \underset{i+1}{;} \gamma)(1 \leq i \leq n - 1)$, while $\omega(\gamma \underset{n}{;} a) = \omega(\gamma)a$

for any $a \in \mathbb{R}_e$ and any $\gamma \in \mathrm{T}^n M$.

(1.5) If γ is an n-microsquare of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\omega(\Sigma_{(i,j)}\gamma) = (-1)^{1+\eta_{i,j}}\omega(\gamma)$ $(1 \leq i < j \leq n)$, where $\eta_{i,j} = \mathbf{p}_i\Sigma_{h=i+1}^{j}\mathbf{p}_h + \mathbf{p}_j\Sigma_{h=i+1}^{j-1}\mathbf{p}_h$.

We denote by $\Omega_n(M)$ the totality of differential n-forms on M. A differential n-form ω on M is called *graded* if it satisfies the following condition:

(1.6) $\omega(\gamma \underset{i}{\cdot} a) = \omega(a \underset{i+1}{\cdot} \gamma)$ $(1 \leq i \leq n-1)$, while $\omega(\gamma \underset{n}{\cdot} a) = \omega(\gamma)a$ for any $a \in \mathbb{R}_o$ and any $\gamma \in \mathrm{T}^n M$.

We denote by $\Omega_n(M)$ the totality of graded differential n-forms on M.

An *n-supermicrocube on M* is a mapping from $D(\mathbf{0}, \mathbf{1})^n$ to M. We denote by $\mathfrak{T}^n M$ the totality of n-supermicrocubes on M. Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $a \in \mathbb{R}$, n-supermicrocubes $\gamma \underset{i}{\cdot} a$ and $a \underset{i}{\cdot} \gamma$ on M $(1 \leq i \leq n)$ are defined as in (1.1) and (1.2), respectively. Given $\overline{\gamma} \in \mathfrak{T}^n M$ and $\sigma \in \mathfrak{Perm}_n$, an n-supermicrocube $\Sigma_\sigma(\overline{\gamma})$ on M is defined as in (1.3). An n-supermicrocube $\overline{\gamma}$ on M is said to be *of type* $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ providing that $\gamma \circ \iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \circ \pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n} = \overline{\gamma}$. A *differential n-superform on M* is a mapping $\overline{\omega}$ from $\mathfrak{T}^n M$ to $\mathbb{R}$ subject to the following conditions:

(1.7) $\overline{\omega}(\overline{\gamma} \underset{i}{\cdot} a = \overline{\omega}(a \underset{i+1}{\cdot} \overline{\gamma})$ $(1 \leq i \leq n-1)$, while $\overline{\omega}(\overline{\gamma} \underset{n}{\cdot} a) = \overline{\omega}(\overline{\gamma})a$ for any $a \in \mathbb{R}$ and any $\overline{\gamma} \in \mathfrak{T}^n M$

(1.8) If $\overline{\gamma}$ is an n-supermicrocube of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\overline{\omega}(\Sigma_{(i,j)}\overline{\gamma}) = (-1)^{1+\eta_{i,j}}\overline{\omega}(\overline{\gamma})(1 \leq i < j \leq n)$, where $\eta_{i,j} = \mathbf{p}_i \, \Sigma_{h=i+1}^{j}\mathbf{p}_h + \mathbf{p}_j \, \Sigma_{h=i+1}^{j-1}\mathbf{p}_h$.

We denote by $\overline{\Omega}_n(M)$ the totality of differential n-superforms on M.

The partial binary operation $\underset{i}{+}$ among $\mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ and among $\mathfrak{T}^n M$ $(1 \leq i \leq n)$ is defined as in Lavendhomme (1996, §3.4, p. 88), for which we have the following additivity of differential n-forms and differential n-superforms:

Proposition 1.1. (1) If ω is a differential n-form and γ and γ' are n-microcubes of the same type such that $\gamma \underset{i}{+} \gamma'$ is defined, then $\omega(\gamma \underset{i}{+} \gamma') = \omega(\gamma) + \omega(\gamma')$ $(1 \leq i \leq n)$.

(2) If $\overline{\omega}$ is a differential n-superform and $\overline{\gamma}$ and $\overline{\gamma}'$ are n-supermicrocubes such that $\overline{\gamma} \underset{i}{+} \overline{\gamma}'$ is defined, then $\overline{\omega}(\overline{\gamma} \underset{i}{+} \overline{\gamma}') = \overline{\omega}(\overline{\gamma}) + \overline{\omega}(\overline{\gamma}')$ $(1 \leq i \leq n)$.

Proof. The first statement follows from the componentwise $\mathbb{R}_e$-homogeneity (1.4) by the same token as in Lavendhomme (1996, §1.2, Proposition 10). Similarly the second statement follows from the componentwise $\mathbb{R}$-homogeneity (1.7). ■

To each $\overline{\omega} \in \Omega_n(M)$ we can assign a graded differential n-form $\Phi(\overline{\omega})$ as follows:

(1.9) If γ is an n-microcube of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, then $\Phi(\overline{\omega})(\gamma) = \overline{\omega}(\gamma \circ \pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n})$.

It is easy to see, by Proposition 1.1, that for any $\overline{\omega} \in \overline{\Omega}_n(M)$ and any $\overline{\gamma} \in \mathfrak{T}^n M$ we have

(1.10) $\overline{\omega}(\overline{\gamma}) = \Sigma_{(\mathbf{p}_1,\ldots,\mathbf{p}_n)\in(\mathbb{Z}_2)^n}\, \overline{\omega}(\overline{\gamma} \circ \iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n} \circ \pi_{\mathbf{p}_1,\ldots,\mathbf{p}_n})$.

Proposition 1.2. For any graded differential n-form ω on M there exists a unique differential n-superform $\overline{\omega}$ on M such that $\omega = \Phi(\overline{\omega})$.

Proof. It follows from (1.9) and (1.10) that if such $\overline{\omega}$ exists, it should be defined as follows:

(1.11) $\overline{\omega}(\overline{\gamma}) = \Sigma_{(\mathbf{p}_1,\ldots,\mathbf{p}_n)\in(\mathbb{Z}_2)^n}\omega(\overline{\gamma} \circ \iota_{\mathbf{p}_1,\ldots,\mathbf{p}_n})$

for any $\overline{\gamma} \in \mathfrak{T}^n M$. By Proposition 1.1 it is easy to see that properties (1.4)–(1.6) imply properties (1.7) and (1.8). ■

The above proposition implies that Φ gives a bijective correspondence between $\overline{\Omega}_n(M)$ and $\Omega_n(M)$.

Now we define the interior product $\mathbf{i}_X\omega$ and the Lie derivative $\mathbf{L}_X\omega$ of a differential n-form ω with respect to a vector field X on M, while exterior differentiation will be discussed in the next section. Recall (Nishimura, 1998) that a vector field on M is a mapping X from $D(\mathbf{0}, \mathbf{1})$ to M^M with X_0 being the identity transformation 1_M of M. The totality of vector fields on M is denoted by $\chi(M)$. It is a $\mathbb{Z}_2$-graded $\mathbb{R}$-bimodule whose even and odd parts $\chi^{\mathbf{0}}(M)$ and $\chi^{\mathbf{1}}(M)$ can naturally be identified with the totality of mappings X from $D^{\mathbf{0}}$ to M^M with $X_0 = 1_M$ and that of mappings Y from D^1 to M^M with $Y_0 = 1_M$, respectively. Given $X \in \chi^{\mathbf{p}_1}(M)$ and $\gamma \in \mathrm{T}^{\mathbf{p}_2,\ldots,\mathbf{p}_n}M$, we define $X * \gamma \in \mathrm{T}^{\mathbf{p}_1\ldots\mathbf{p}_n}M$ as follows:

(1.12) $(X * \gamma)(d_1, \ldots, d_n) = X_{d_1}(\gamma(d_2, \ldots, d_n))$

for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$. Given $X \in \chi^{\mathbf{p}}(M)$ and $\omega \in \Omega_n(M)$, we define $\mathbf{i}_X\omega \in \Omega_{n-1}(M)$ as follows:

(1.13) $(\mathbf{i}_X\omega)(\gamma) = \omega(X * \gamma)$ for any $\gamma \in \mathrm{T}^{n-1}M$

The $\mathbb{R}$-module $\Omega_n(M)$ is Euclidean, so that given $X \in \chi^{\mathbf{p}}(M)$ and $\omega \in \Omega_n(M)$, we can define $\mathbf{L}_X\omega \in \Omega_n(M)$ to be unique such that

(1.14) $(X_d)^*\omega - \omega = d\mathbf{L}_X\omega$ for any $d \in D^{\mathbf{p}}$

where $((X_d)^*\omega)(\gamma) = \omega(X_d \circ \gamma)$.

For $X \in \chi(M)$ we define $\mathbf{i}_X\omega$ and $\mathbf{L}_X\omega$ as follows:

(1.15) $\mathbf{i}_X\omega = \mathbf{i}_{X_e}\omega + \mathbf{i}_{X_o}\omega$

(1.16) $\mathbf{L}_X\omega = \mathbf{L}_{X_e}\omega + \mathbf{L}_{X_o}\omega$

It is easy to see the following result.

Proposition 1.3. If X is a vector field on M and ω is a graded differential n-form on M, then differential forms $\mathbf{i}_X\omega$ and $\mathbf{L}_X\omega$ are graded.

Proposition 1.4. For any $X \in \chi^{\mathbf{p}}(M)$ and $Y \in \chi^{\mathbf{q}}(M)$ we have

$$(1.17) \quad \mathbf{L}_{[X,Y]} = \mathbf{L}_X\mathbf{L}_Y - (-1)^{\mathbf{pq}}\mathbf{L}_Y\mathbf{L}_X$$
$$(1.18) \quad \mathbf{i}_{[X,Y]} = \mathbf{L}_X\mathbf{i}_Y - (-1)^{\mathbf{pq}}\mathbf{i}_Y\mathbf{L}_X$$

Proof. (1.17) follows by the same token as in Nishimura (1997a, Theorem 1.6), while (1.18) follows by the same token as in Nishimura (1997a, Theorem 1.8). ■

2. EXTERIOR DIFFERENTIATION

A *marked n-microcube of type* $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M is a pair $(\gamma, \underline{e})$ of an n-microcube γ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M and $\underline{e} = (e_1, \ldots, e_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$. We denote by $\tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ the totality of marked n-microcubes of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M. We denote by $\tilde{\mathbf{T}}^n M$ the set-theoretic union of $\tilde{\mathbf{T}}^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$ for all $(\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{Z}_2)^n$. We denote by $\mathbf{C}_n(M)$ the free $\mathbb{R}$-module generated by $\tilde{\mathbf{T}}^n M$. The boundary operator ∂: $\mathbf{C}_{n+1}(M) \to \mathbf{C}_n(M)$ is defined on the generators of $\mathbf{C}_{n+1}(M)$ as follows and extended to $\mathbf{C}_{n+1}(M)$ by $\mathbb{R}$-linearity:

$$(2.1) \quad \partial\,(\gamma, \underline{e}) = \sum_{i=1}^{n+1} (-1)^{i+\alpha_i} G^i(\gamma, \underline{e})$$

where

$$(2.2) \quad \alpha_i = \mathbf{p}_i\Big(\sum_{h\neq i} \mathbf{p}_h\Big)$$

(2.3) $G^i(\gamma, \underline{e}) = F^i_0(\gamma, \underline{e}) - F^i_1(\gamma, \underline{e})$ with $F^i_j(\gamma, \underline{e}) = (\gamma^i_j, (e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}))$ and $\gamma^i_j(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, je_i, d_i, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \cdots \times D^{\mathbf{p}_{n+1}}$ $(j = 0, 1)$

Proposition 2.1. $\partial \circ \partial = 0$.

Proof. Let (γ, e) be a marked $(n + 1)$-microcube of type $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1})$ on M. For any natural numbers i, j with $1 \leq i < j \leq n + 1$, the coefficient of $G^i(G^j(\gamma, \underline{e}))$ in $\partial(\partial(\gamma, \underline{e}))$ is $(-1)^{i+j+\alpha_j+\beta_i}$ with $\alpha_j = \mathbf{p}_j(\Sigma_{h\neq j}\, \mathbf{p}_h)$ and $\beta_i = \mathbf{p}_i(\Sigma_{h\neq i,j}\, \mathbf{p}_h)$, while the coefficient of $G^{j-1}(G^i(\gamma, \underline{e}))$ in $\partial(\partial(\gamma, \underline{e}))$ is $(-1)^{i+j-1+\alpha_i+\beta_j}$ with $\alpha_i = \mathbf{p}_i(\Sigma_{h\neq i}\, \mathbf{p}_h)$ and $\beta_j = \mathbf{p}_j(\Sigma_{h\neq i,j}\, \mathbf{p}_h)$. Since $\alpha_j + \beta_i = \alpha_i + \beta_j = \mathbf{p}_i\mathbf{p}_j$, the desired statement obtains by the same token as in Lavendhomme (1996, §4.2, Proposition 1). ■

Given $(\gamma, \underline{e}) \in \tilde{\mathrm{T}}^n M$ and $\omega \in \Omega_n(M)$ with $e = (e_1, \ldots, e_n)$, the *integral* $\int_{(\gamma,\underline{e})}\omega$ of ω on $(\gamma, \underline{e})$ is defined as follows:

$$(2.4)\quad \int_{(\gamma,\underline{e})} \omega = \omega\,(\gamma)e_1 \ldots e_n$$

The integral can be extended linearly to $\mathbf{C}_n(M)$.

Given $\omega \in \Omega_n(M)$, it is easy to see that the function $\varphi\colon \tilde{\mathrm{T}}^n(M) \to \mathbb{R}$ defined by $\varphi(\gamma, \underline{e}) = \int_{(\gamma,e)} \omega$ satisfies the following properties:

(2.5) For each $\underline{e}$ in $D^{(\mathbf{p}_1,\ldots,\mathbf{p}_n)}$, $\varphi(\cdot,\ e)$ is n-$\mathbb{R}_e$-homogeneous in $T^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$.

(2.6) For each γ in $T^{\mathbf{p}_1,\ldots,\mathbf{p}_n}M$, $\varphi(\gamma,\ \cdot)$ is n-$\mathbb{R}_e$-homogeneous in $D^{(\mathbf{p}_1,\ldots,\mathbf{p}_n)}$.

(2.7) For each σ in $\mathfrak{Perm}_n$ and each marked n-microcube $(\gamma, \underline{e})$ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ on M, we have $\varphi(\Sigma_\sigma(\gamma)), e^{\sigma-1}) = |\sigma|\varphi(\gamma, \underline{e})$ where $|\sigma|$ is the sign of σ.

Now we have the following converse.

Proposition 2.2. If a function φ from $\tilde{\mathrm{T}}^n$M to $\mathbb{R}$ obeys conditions (2.5)–(2.7), then there exists a unique differential n-form ω on M such that

$$(2.8)\quad \varphi(\gamma, \underline{e}) = \int_{(\gamma,e)} \omega$$

Proof. By the same token as in Lavendhomme (1996, §4.2, Proposition 2). ■

Proposition 2.3. For any differential n-form ω on M there exists a unique differential $(n+1)$-form $\mathrm{d}\omega$ such that

$$(2.9)\quad \int_{\partial(\gamma,\underline{e})} \omega = \int_{(\gamma,\underline{e})} \mathrm{d}\omega$$

for any $(\gamma, \underline{e})$ in $\tilde{\mathrm{T}}^n M$.

Proof. This follows from Proposition 2.2 by the same token as in Lavendhomme (1996, §4.2, Proposition 3). ■

Proposition 2.4. We have

$$(2.10)\quad \mathrm{d}\omega(\gamma) = \sum_{i=1}^{n+1} (-1)^{i+1+\xi_i}(F^i\overleftarrow{\mathbf{D}}_{\mathbf{p}_i})(0)$$

for any n-microcube γ of type $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1})$ on M, where $\xi_i = \mathbf{p}_i\,\Sigma_{h>i}\,\mathbf{p}_h$ and $F^i(e) = \omega(\gamma^i(e))$ for any $e \in D^{\mathbf{p}_i}$ with $\gamma^i(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots,$

$d_{i-1}, e, d_i, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \cdots \times D^{\mathbf{p}_{n+1}}$.

Proof. This follows from the proof of Proposition 2.3 by the same token as in Lavendhomme (1996, §4.2, Proposition 4). ■

Proposition 2.5. If a differential n-form ω on M is graded, so is $\mathrm{d}\omega$.

The above proposition follows from the following two lemmas.

Lemma 2.6. We have

$$\text{(2.11)} \quad \mathrm{d}\omega(\gamma \underset{n+1}{\dot{+}} a) = \mathrm{d}\omega(\gamma)a$$

for any $(\mathbf{p}_1, \ldots, \mathbf{p}_{n+1}) \in (\mathbb{Z}_2)^{n+1}$, any $\gamma \in \mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, and any $a \in \mathbb{R}_{\mathrm{o}}$.

Proof. Let $(\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{p}_{n+1}) = (\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{p}_{n+1} + 1)$. By Proposition 2.4 we have

$$\text{(2.12)} \quad \mathrm{d}\omega(\gamma \underset{n+1}{\dot{+}} a) = \sum_{i=1}^{n+1} (-1)^{i+1+\xi_i}(F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i})(0)$$

$$\text{(2.13)} \quad \mathrm{d}\omega(\gamma) = \sum_{i=1}^{n+1} (-1)^{i+1+\xi_i}(F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i})(0)$$

where $\underline{\xi}_i = \underline{\mathbf{p}}_i \, \Sigma_{h>i} \, \underline{\mathbf{p}}_h$ and $\underline{F}^i(e) = \omega(\underline{\gamma}^i(e))$ for any $e \in D^{\mathbf{p}_i}$ with $\underline{\gamma}^i(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_n a)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \cdots \times D^{\mathbf{p}_{n+1}}$ $(1 \le i \le n)$ and $\underline{\gamma}^{n+1}(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_n, ae)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_n}$, while $\xi_i = \mathbf{p}_i \, \Sigma_{h>i} \, \mathbf{p}_h$ and $F^i(e) = \omega(\gamma^i(e))$ for any $e \in D^{\mathbf{p}_i}$ with $\gamma^i(e)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\mathbf{p}_1} \times \cdots \times D^{\mathbf{p}_{i-1}} \times D^{\mathbf{p}_{i+1}} \times \cdots \times D^{\mathbf{p}_{n+1}}$ $(1 \le i \le n+1)$. For any natural number i with $1 \le i \le n$ and any $e \in D^{\mathbf{p}_i}$ we have

$$\text{(2.14)} \quad \begin{aligned} \underline{F}^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)e &= \underline{F}^i(e) - \underline{F}^i(0) \\ &= (F^i(e) - F^i(0))a \qquad \text{[since } \omega \text{ is graded]} \\ &= F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ea \\ &= (-1)^{\mathbf{p}_i} F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ea. \end{aligned}$$

so that

$$\text{(2.15)} \quad \underline{F}^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0) = (-1)^{\mathbf{p}_i} F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)a$$

On the other hand, for any $e \in D^{\mathbf{p}_{n+1}}$, we have

$$(2.16)\quad \underline{F}^{n+1}\overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_{n+1}}(0) = F^{n+1}\overleftarrow{\mathbf{D}}_{\mathbf{p}_{n+1}}(0)a$$

Since $\xi_i = \xi_i + \mathbf{p}_i$ $(1 \leq i \leq n)$ and $\xi_{n+1} = \xi_{n+1} = 0$, the desired equality (2.11) follows from (2.12), (2.13), (2.15), and (2.16). ■

Lemma 2.7. We have

$$(2.17)\quad d\omega(\gamma \underset{i}{\cdot} a) = d\omega(a \underset{i+1}{\cdot} \gamma) \qquad (1 \leq i \leq n)$$

for any $(p_1, \ldots, p_{n+1}) \in (\mathbb{Z}_2)^{n+1}$, any $\gamma \in \mathrm{T}^{\mathbf{p}_1,\ldots,\mathbf{p}_{n+1}}M$, and any $a \in \mathbb{R}_0$.

Proof. Let $(\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_i, \ldots, \underline{\mathbf{p}}_{n+1}) = (\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_i + 1, \ldots, \underline{\mathbf{p}}_{n+1})$ and $(\overline{\mathbf{p}}_1, \ldots, \overline{\mathbf{p}}_{i+1}, \ldots, \overline{\mathbf{p}}_{n+1}) = (\mathbf{p}_1, \ldots, \mathbf{p}_{i+1} + 1, \ldots, \mathbf{p}_{n+1})$. Let ξ_j and F^j $(1 \leq j \leq n+1)$ be the same as in Proposition 2.4. By Proposition 2.4 we have

$$(2.18)\quad d\omega(\gamma \underset{i}{\cdot} a) = \sum_{j=1}^{n+1} (-1)^{j+1+\underline{\xi}_j}(\underline{F}^j\overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_j})(0)$$

$$(2.19)\quad d\omega(a \underset{i+1}{\cdot} \gamma) = \sum_{j=1}^{n+1} (-1)^{j+1+\overline{\xi}_j}(\overline{F}^j\overleftarrow{\mathbf{D}}_{\overline{\mathbf{p}}_j})(0)$$

where $\underline{\xi}_j = \underline{\mathbf{p}}_j \sum_{h>j} \underline{\mathbf{p}}_h$ and $\underline{F}^j(e) = \omega(\underline{\gamma}^j(e))$ for any $e \in D^{\underline{\mathbf{p}}_j}$ with $\underline{\gamma}^j(e)(d_1, \ldots, d_n) = (\gamma \underset{i}{\cdot} a)(d_1, \ldots, d_{j-1}, e, d_j, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\underline{\mathbf{p}}_1} \times \cdots \times D^{\underline{\mathbf{p}}_{j-1}} \times D^{\underline{\mathbf{p}}_{j+1}} \times \cdots \times D^{\underline{\mathbf{p}}_{n+1}}$ $(1 \leq j \leq n+1)$, while $\overline{\xi}_j = \overline{\mathbf{p}}_j \sum_{h>j} \overline{\mathbf{p}}_h$ and $\overline{F}^j(e) = \omega(\overline{\gamma}^j(e))$ for any $e \in D^{\overline{\mathbf{p}}_j}$ with $\overline{\gamma}^j(e)(d_1, \ldots, d_n) = (a \underset{i+1}{\cdot} \gamma)(d_1, \ldots, d_{j-1}, e, d_j, \ldots, d_n)$ for any $(d_1, \ldots, d_n) \in D^{\overline{\mathbf{p}}_1} \times \cdots \times D^{\overline{\mathbf{p}}_{j-1}} \times D^{\overline{\mathbf{p}}_{j+1}} \times \cdots \times D^{\overline{\mathbf{p}}_{n+1}}$ $(1 \leq j \leq n+1)$. For any j with $j \neq i$ and $j \neq i+1$,

$$\begin{aligned}(2.20)\quad \underline{F}^j\overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)e &= \underline{F}^j(e) - \underline{F}^j(0) \\ &= F^j(e) - F^j(0) \qquad \text{[since } \omega \text{ is graded]} \\ &= F^j\overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)e\end{aligned}$$

so that

$$(2.21)\quad \underline{F}^j\overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0) = F^j\overleftarrow{\mathbf{D}}_{\mathbf{p}_j}(0)$$

For $j = i$ we have that for any $e \in D^{\mathbf{p}_i}$,

$$(2.22)\quad \underline{F}^i \overleftarrow{\mathbf{D}}_{\underline{\mathbf{p}}_i}(0)e = F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ae$$

$$= (-1)^{\mathbf{p}_i} F^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0)ea$$

$$= (-1)^{\mathbf{p}_i}\{F^i(e) - F^i(0)\}a$$

$$= (-1)^{\Sigma_{h\geq i}\mathbf{p}_h}\{\bar{F}^i(e) - \bar{F}^i(0)\}$$

[since ω is graded]

$$= (-1)^{\Sigma_{h\geq i}\mathbf{p}_h} \bar{F}^i \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_i}(0)e$$

so that

$$(2.23)\quad \underline{F}^i \overleftarrow{\mathbf{D}}_{\mathbf{p}_i}(0) = (-1)^{\Sigma_{h\geq i}\mathbf{p}_h} \bar{F}^i \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_i}(0)$$

For $j = i + 1$ we have that for any $e \in D^{\bar{\mathbf{p}}_{i+1}}$,

$$(2.24)\quad \bar{F}^{i+1} \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_{i+1}}(0)e = (-1)^{\bar{\mathbf{p}}_{i+1}} F^{i+1} \overleftarrow{\mathbf{D}}_{\mathbf{p}_{i+1}}(0)ae$$

$$= F^{i+1} \overleftarrow{\mathbf{D}}_{\mathbf{p}_{i+1}}(0)ea$$

$$= \{F^{i+1}(e) - F^{i+1}(0)\}a$$

$$= (-1)^{\Sigma_{h>i+1}\mathbf{p}_h}\{F^{i+1}(e) - F^{i+1}(0)\}$$

[since ω is graded]

$$= (-1)^{\Sigma_{h>i+1}\mathbf{p}_h} F^i \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_i}(0)e$$

so that

$$(2.25)\quad \bar{F}^{i+1} \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_{i+1}}(0) = (-1)^{\Sigma_{h>i+1}\mathbf{p}_h} F^i \overleftarrow{\mathbf{D}}_{\bar{\mathbf{p}}_i}(0)$$

For j with $j < i$ we have

$$(2.26)\quad \underline{\xi}_j = \bar{\xi}_j = \mathbf{p}_j + \xi_j$$

while for j with $j > i + 1$ we have

$$(2.27)\quad \underline{\xi}_j = \bar{\xi}_j = \xi_j$$

On the other hand, we have

$$(2.28)\quad \underline{\xi}_i = \xi_i + \sum_{h>i} \mathbf{p}_h$$

$$= \bar{\xi}_i + \sum_{h\geq i} \mathbf{p}_h$$

$$(2.29)\quad \bar{\xi}_{i+1} = \xi_{i+1} + \sum_{h>i+1} \mathbf{p}_h$$

$$= \xi_{i+1} + \sum_{h>i+1} \mathbf{p}_h$$

Therefore (2.17) follows from (2.18), (2.19), (2.21), (2.23), (2.25), and (2.26)–(2.29). ■

We conclude this section with the remaining one of Cartan's three magic formulas in our super context.

Proposition 2.8. For any $X \in \chi(M)$ we have

$$(2.30)\quad \mathbf{L}_X = d\mathbf{i}_X + \mathbf{i}_X d$$

Proof. This follows from Proposition 2.4 by the same token as in Nishimura (1997a, Theorem 1.9). ■

3. DE RHAM's THEOREM

In this section we establish a cubical, infinitesimal, and super version of de Rham's theorem on the level of chain complexes after the manner of Félix and Lavendhomme (1990). Our result is more frivolous than theirs, but we could not expect much more, because, as is well known, Stokes' theorem does not obtain in our super context.

Proposition 3.1. We have

$$(3.1)\quad \mathbf{d} \circ \mathbf{d} = 0$$

so that $(\Omega_n(M), \mathbf{d})$ is a chain complex.

Proof. This follows from Proposition 2.1. ■

We say that two elements c and c' of $\mathbf{C}_n(M)$ are *equivalent*, in notation $c \sim c'$, provided that $\int_c \omega = \int_{c'} \omega$ for any $\omega \in \Omega_n(M)$. Obviously the relation $\sim$ is an equivalence relation, with respect to which the quotient module of $\mathbf{C}_n(M)$ is denoted by $\mathrm{S}_n(M)$. It is easy to see that for any $c \in \mathbf{C}_n(M)$, $c \sim 0$ implies $\partial c \sim 0$, so that we have a chain complex $(\mathrm{S}_n(M), \partial)$, whose dual chain complex is denoted by $(\mathrm{S}^n(M), \delta)$.

Proposition 3.2. For every n the diagram

$$\begin{tikzcd} \Omega_n(M) \arrow[r, "\int"] \arrow[d, "\mathbf{d}"'] & \mathrm{S}^n(M) \arrow[d, "\delta"] \\ \omega_{n+1}(M) \arrow[r, "\int"'] & \mathrm{S}^{n+1}(M) \end{tikzcd}$$

commutes, so that $\int$ is a morphism of chain complexes.

Proof. This follows from the definition of $\mathbf{d}$. ■

Theorem 3.3. The morphism $\int$ of chain complexes from $(\Omega_n(M), \mathbf{d})$ to $(\mathrm{S}^n(M), \delta)$ is an isomorphism, so that the two chain complexes are isomorphic.

Proof. This follows from Proposition 2.2. ■

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